Unravelling Chaos : The Lorenz Attractor

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1 Introduction

1.1 What is chaos theory ?

Chaos theory is a branch of mathematics and physics that studies the behavior of dynamical systems that are:

- highly sensitive to initial conditions.
- unpredictable, even though they are deterministic.
- non-periodic

Example: double pendulum.

Simply said : Chaos theory is the study of how small changes in the initial conditions of a system can lead to large and unpredictable changes in the system's behavior over time.

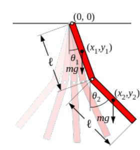

Figure 1: Double pendulum

1.2 Why is it important?

Chaos theory has many applications in engineering, including:

- Control systems: Chaos theory can be used to design more robust and reliable control systems.For example, a chaos-based controller can be used to keep a robot upright even if it is walking on uneven terrain.
- Robotics: Chaos theory can be used to design more agile and responsive robots. For example, a chaos-based controller can be used to control a robot's arm so that it can quickly and accurately reach for objects.
- Signal processing: Chaos theory can be used to develop new signal processing techniques for filtering and noise reduction. a chaos-based filter can be used to remove noise from a signal without distorting the signal itself.
- Financial modeling: Chaos theory can be used to develop more accurate models of financial markets. a chaos-based model can be used to predict the future value of a stock or the risk of a financial crisis.
- Meteorology.

2 Brief Overview of the Lorenz attractor

2.1 History

- Discovered in 1963 by Edward Lorenz american mathematician and meteorologist, while studying a simplified model of atmospheric convection, seeking to improve weather predictions.
- While working on this model on his computer, he noticed that small changes lead to vastly different outcomes. Thus the term "Butterfly effect" was born.
- The Lorenz Attractor quickly became a symbol of deterministic chaos and is one of the most recognizable examples in chaos theory.

2.2 Mathematical description

Lorenz attractor is a strange attractor that arises from the Lorenz equations, a system of three ordinary differential equations that describe the evolution of 3 variables x,y and z.

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dy}{dt} = x(\rho - z) - y \\ \frac{dz}{dt} = xy - \beta \end{cases} . \tag{1}$$

- x, y and z are the three state variables that describe the states of the system. t represents time.
- σ represents fluid viscosity, Prandtl number.
- ρ is a positive constant, it is the heating parameter.
- β : aspect ratio, influences scale of the system.

2.3 Visual representation

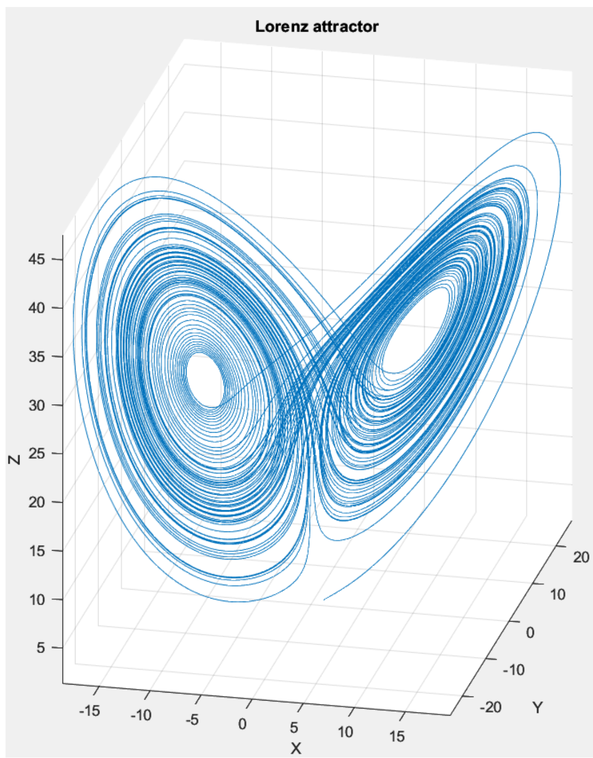

Figure 2: Lorenz attractor visualized in MatLab, obtained with values $\sigma = 10$, $\rho = 28$, β=8/3

3 What is the Lorenz attractor ?

Definition(Dynamical System): "A system in which a function describes the time dependence of a point in an ambient space." Simply said: a particle or ensemble whose state varies over time and obeys differential equations which have time derivatives

In some cases, the states in the system evolve towards a specific set of states, for a wide variety of initial states. In dynamical systems, that is an **attractor**. Depending on the dimensions, an attractor can be a point, a finite set of points, a curve, a manifold, or even a complicated set with a fractal structure. Example : Damped pendulum. Attractor is caused by dissipation.

3.1 Attractors

Fixed Point (Steady-State) Attractor:

- simplest type of attractor
- the system converges to a single stable state over time.
- represents equilibrium, where the system settles at a particular point and remains there.

Example of a fixed point attractor is a pendulum coming to rest at the bottom of its swing.

Limit Cycle (Periodic) Attractor:

- system repeats a specific trajectory or behavior over time.
- often periodic

Example : ideal oscillators.

Strange attractor :

- Non-periodic, complex behavior
- Fractal Structure
- Self Similarity
- Infinite number of points
- Structure and Boundedness

4 Study : Is weather chaotic ?

Study by :
Shen, Bo-Wen, R. A. Pielke Sr., X. Zeng, J.-J. Baik, S. Faghih-Naini, J. Cui, R. Atlas, et T. A. L. Reyes. "Is Weather Chaotic? Coexisting Chaotic and Non-chaotic Attractors Within Lorenz Models". In 13th Chaotic Modeling and Simulation International Conference, édité par Christos H. Skiadas et Yiannis Dimotikalis, 805-25. Cham: Springer International Publishing, 2021.

The aim of this study was to answer the question, by 2 main points, first illustrate two kinds of attractors coexistence within Lorenz models (same model but different initial conditions). Second is Suggest that the entirety of weather possesses the dual nature of chaos and order associated with chaotic and non-Chaotic processes.

$$\frac{dX}{d\tau} = \sigma Y - \sigma X,$$

$$\frac{dY}{d\tau} = -XZ + rX - Y,$$

$$\frac{dZ}{d\tau} = XY - bZ.$$

Figure 3: Lorenz model used in the study

The first and last regime are insensitive to ICs whereas as we see in the middle, within the chaotic regime, two solution orbits whose starting points are very close to each other display very different time evolutions, as clearly shown in blue and red in Fig. 1e. The phenomenon is called the sensitive dependence of solutions on ICs and only appears within a chaotic solution. Other than these different types of solutions they studied how the butterfly pattern is crucial in proving the existence of two coexisting attractors.

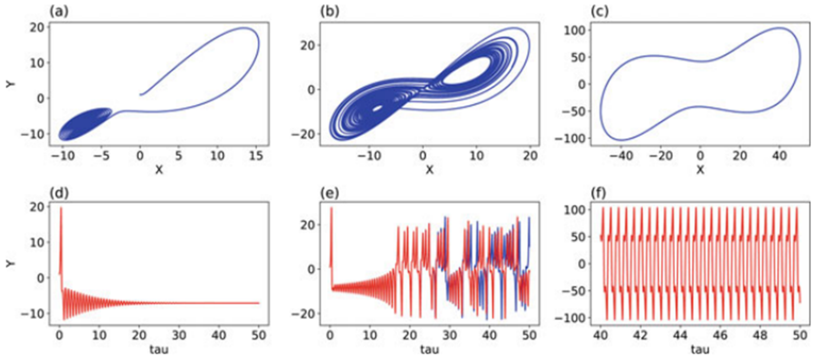

Figure 4: Three types of solutions within the 3DLM. Left, middle, and right panels displays steadystate, chaotic, and limit cycle solutions at small, moderate, and large heating parameters (i.e.,r = 20, 28, and 350), respectively. The solutions are categorized into a point attractor, a chaotic attractor, and a periodic attractor.

4.1 Boundedness and Divergence of Chaotic Solutions

When studying chaotic systems like this one we are often in interested what is called the lyapunov exponent. The Lyapunov exponent (LE) measures how quickly nearby trajectories diverge. So we have two nearby but different solutions and we see over time how much they diverge from each other.

- If $\lambda > 0$: Exponential rate in averaged separation.
- If $\lambda = 0$: Means two trajectories stay parallel to each other
- If $\lambda < 0$: Means average separation will converge exponentially to 0.

Chaotic solutions have a positive Lyapunov exponent.

Why is boundedness important in this study?

In chaotic systems, despite the apparent unpredictability, the solutions remain confined within certain boundaries due to the bounded attractors. This is crucial in showing that chaos can coexist with other attractors because it implies that, within the system's overall complexity, there are regions of stability and predictability.

1st Simulation of the 3DLM with parameters r = 24.4, σ = 10, and b = 8/3, but different initial conditions.

Result : Two types of solutions that include steady state and chaotic solutions.

2nd Simulation of the 3DLM:

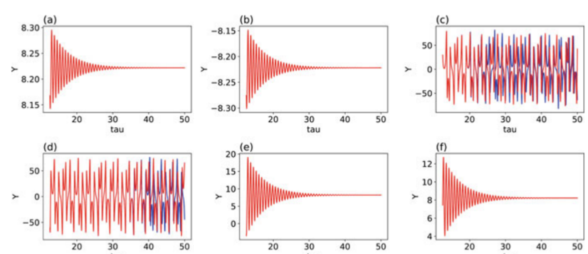

Figure 5: A co-existence of chaotic (c, d) and non-chaotic (a, b, e, f) solutions using the same parameters for σ = 1, b = 0.4, and r = 170 within the 3DLM

4.2 Results

"our results show that chaotic and non-chaotic solutions may coexist [...] The analysis suggests a need to refine our view of weather by taking the dual nature associated with attractor coexistence into consideration. To this end, we suggest, contrary to the traditional view that weather is chaotic, that weather is, in fact, a superset that consists of both chaotic and non-chaotic processes, including both order and chaos."

X	Y	Z
X_c	$Y_c + 1$	Z_c
$-X_c$	$-Y_c + 1$	Z_c
0	1	0
−76.72346293	37.62433028	−146.96230812
−27.75526885	167.67883615	3.66782724
136.44623635	99.45689394	−19.76741851

Figure 6: Initial conditions used for revealing the coexistence of two attractors

5 Youtube videos

https://www.youtube.com/watch?v=uzJXeluCKMs