2. Propositional fuzzy logic

2.1 Classical and Fuzzy Implication and Inference

In classical propositional logic we use mainly two formal constructions: ***Implication*** (sometimes named as „conditional statement“) and ***Inference*** (logical derivation, deduction). Construction of Inference is sometimes called as rule „ Modus ponens.

$p \Rightarrow q$

p	q	$\Rightarrow$
0	0	1*
0	1	1*
1	0	0
1	1	1

a) Implication

$(p, (p \Rightarrow q)) \Rightarrow q$

p	$p \Rightarrow q$	q
0	0	1*
0	1	1*
1	0	0
1	1	1

b) Inference

Fig. 6.4.

Example 1:

p … x is a thermodynamic machine, q … x generates a causal time,

Human organism is not a thermodynamic machine.

Incomplete conversion.

In fuzzy logic we recognize also such a Incomplete conversion.

The way to implication and to inference explained above goes from classical logic as examples of „transfer of truth”.

The tables 6.4.a) and 6.4.b) are transformed in fuzzy logic in tables Fig. 7.4.

Fuzzy implication

(p, μ(p))	(q, μ(q))	($\Rightarrow$, μ($\Rightarrow$))
0.1	0.1	near to 1*
0.1	0.8	near to 1*
0.9	0.1	near to 0
0.9	0.9	near to 1

Fuzzy inference

(p, μ(p))	($\Rightarrow$, μ($\Rightarrow$))	(q, μ(q))
0.1	0.1	near to 1*
0.1	0.8	near to 1*
0.9	0.1	near to 0
1	1	near to 1

Fig. 7.4.

Algorithms for computation of values of membership functions for implication and inference are various and are named according to their authors:

a) Computation of values of membership functions for implication and inference according to *Mamdani*:

$$\mu_{IMM}(\Rightarrow) = \min\{\mu(p), \mu(q)\}. \tag{20.4}$$

$$\mu_{InfM}(q) = \begin{cases} \mu(p), \text{ for } \mu_{IMM}(\Rightarrow) \geq \mu(p), \\ \mu_{IMM}(\Rightarrow), \text{ for } \mu_{IMM}(\Rightarrow) < \mu(p). \end{cases} \tag{21.4}$$

Fuzzy implication

$(p, \mu(p))$	$(q, \mu(q))$	$(\Rightarrow, \mu(\Rightarrow))$	$(\Rightarrow, \mu(\Rightarrow))$
0.1	0.1	near to 1*	0.1
0.1	0.8	near to 1*	0.1
0.9	0.1	near to 0	0.1
0.9	0.9	near to 1	0.9

Fuzzy inference

$(p, \mu(p))$	$(\Rightarrow, \mu(\Rightarrow))$	$(q, \mu(q))$	$(q, \mu(q))$
0.1	0.1	near to 1*	0.1
0.1	0.8	near to 1*	0.1
0.9	0.1	near to 0	0,1
0.9	0.9	near to 1	0.9

Fig. 8.4.

Example 2:

$\mu(p) = 0.5$, $\mu(q) = 0.2$, then $\mu(\Rightarrow) = 0.2$

When $\mu(p) = 0.5$ and $\mu(\Rightarrow) = 0.9 \Rightarrow \mu(q) = 0.5$

b) Computation of values of membership functions for implication and inference according to *Lukasiewicz*:

$$\mu_{IMLu}(\Rightarrow) = \min\{1, (1 - \mu(p) + \mu(q))\}, \tag{22.4}$$

$$\mu_{InfLu}(q) = \max\{0, (\mu(p) + \mu_{IMLu}(\Rightarrow) - 1)\}. \tag{23.4}$$

Fuzzy implication

$(p, \mu(p))$	$(q, \mu(q))$	$(\Rightarrow, \mu(\Rightarrow))$	$(\Rightarrow, \mu(\Rightarrow))$
0.1	0.1	near to 1*	1
0.1	0.8	near to 1*	1
0.9	0.1	near to 0	0.2
0.9	0.9	near to 1	1

Fuzzy inference

$(p, \mu(p))$	$(\Rightarrow, \mu(\Rightarrow))$	$(q, \mu(q))$	$(q, \mu(q))$
0.1	0.1	near to 1*	0
0.1	0.8	near to 1*	0
0.9	0.1	near to 0	0
0.9	0.9	near to 1	0.8

Fig. 9.4.

Example 3:

$\mu(p) = 0.5$, $\mu(q) = 0.2$ $\mu(\Rightarrow) = 0.7$

When $\mu(p) = 0.5$ and $\mu(\Rightarrow) = 0.9$ $\Rightarrow$ $\mu(q) = 0.4$

c) Computation of values of membership functions for implication and inference according to ***Larsen***:

$$\mu_{IMLa}(\Rightarrow) = \mu(p) * \mu(q). \tag{24.4}$$

$$\mu_{InfLa}(y) = \begin{cases} \mu(p), \text{ for } \mu_{IMLa}(\Rightarrow) \geq \mu(p), \\ \mu_{IMLa}(\Rightarrow), \text{ for } \mu_{IMLa}(\Rightarrow) < \mu(p). \end{cases} \tag{25.4}$$

Fuzzy implication

(p, μ(p))	(q, μ(q))	($\Rightarrow$, $\mu(\Rightarrow)$)	($\Rightarrow$, $\mu(\Rightarrow)$)
0.1	0.1	Near to 1*	0.01
0.1	0.8	Near to 1*	0.08
0.9	0.1	Near to 0	0.09
0.9	0.9	Near to 1	0.81

Fuzzy inference

(p, μ(p))	($\Rightarrow$, $\mu(\Rightarrow)$)	(q, μ(q))	(q, μ(q))
0.1	0.1	Near to 1*	0.1
0.1	0.8	Near to 1*	0.1
0.9	0.1	near to 0	0.1
0.9	0.9	near to 1	0.9

Fig. 10.4.

Example 4:

$\mu(p) = 0.5$, $\mu(q) = 0.2$ $\mu(\Rightarrow) = 0.01$

When $\mu(p) = 0.5$ and $\mu(\Rightarrow) = 0.9$ $\Rightarrow$ $\mu(q) = 0.5$

2.2 Composition rule

However – exists another way to fuzzy implication and fuzzy inference. Statements as (p, q, $p \Rightarrow q$, …) could be understood as sets (or fuzzy sets, e.g., p: temperature is high, q: consumption is acceptable, …,) or relations (or fuzzy relations,e.g., $p \Rightarrow q$: „if temperature is high“ $\Rightarrow$ „consumption is acceptable“) and relation of inference is possible to express as a relational product:

$$„\ p \circ (p \Rightarrow q) = q\ „$$

In case that „p“ and „$p \Rightarrow q$“ have values of membership functions ((p, $\mu(p)$), $\mu(\Rightarrow)$) we are interested in the computation of the membership function value of the conclusion $\mu(q)$.

The formula for this computation is called ***fuzzy composition rule.***

Definition 1.2: Let us consider fuzzy sets A, B on universes X, Y and binary relation R on Catersian product $X \times Y$.

Fuzzy composition rule enables to construct fuzzy set B as a composition (relation product) of fuzzy set A and fuzzy relation R :

$$B = A \circ R = (R \circ A) \qquad (18.4)$$

In case of finite universes X, Y the relation (18.4) has a form

$$\mu_{AoR}(y) = \mu_B(y) = \max_{\forall x \in X} \{\min \{\mu_A(x), \mu_R(x, y)\}\}, \text{ pro } y \in Y. \qquad (19.4)$$

Relation (19.4) represents a procedure for the computation of the value of membership function to fuzzy set B for (only one !!) y from the set Y. If we want to compute values of membership function for all elements from Y we have to repeat the computation (19.4) for each element from Y.

In case that R is a *fuzzy implication* fuzzy composition rule realizes ***fuzzy inference*** .

***Example 5*: **Let us consider a case of simple car diagnostics. Diagnosis we execute by means of observable symptoms (X) and their available realization (fuzzy set **A)** and by means of known fuzzy implication **R** - (that assigns symptoms to possible faults Y with uncertainties (fuzzy set **B**)).

Symptoms = X = {1... The car does not start, 2 ... The engine is very quickly overheated, 3 ... The power of engine is decreasing during 30 minutes, 4 ... The sound of the engine is too noisy (impacts of valves are heard),
Faults = Y = {5 ... Weak power of accumulator, 6 ... Wrong composition of the fuel, 7 ... Small volume of the oil, 8 ... violated clutch}.

Fuzzy implication **R** is represented by matrix **IM**:

IM	5	6	7	8
1	1	0.5	0.3	0.0
2	0.0	0.5	0.8	0.2
3	0.0	0.6	0.2	0.8
4	0.0	0.8	0.4	0.0

Realization of symptoms: **A** = {0.5/1, 0.5/2, 0.8/3, 0.2/4}.

Compute:

a) fuzzy set **B** as a consequence of action of realization **A**,

b) values of membership functions of faults as consequence of symptom x = 2 (other symptoms are "invisible").

a) **B** = {0.5/1, 0.5/2, 0.8/3, 0.2/4} **o IM** =

= {0.5/1, 0.5/2, 0.8/3, 0.2/4} **o**

IM	5	6	7	8
1	1	0.5	0.3	0.0
2	0.0	0.5	0.8	0.2
3	0.0	0.6	0.2	0.8
4	0.0	0.8	0.4	0.0

= { μ_B (5)=max { min {0.5, 1}, min {0.5, 0.0}, min {0.8, 0.0}, min {0.2, 0.0}},

μ_B (6)=max { min {0.5 ,0.5}, min {0.5, 0.5}, min {0.8, 0.6}, min {0.2, 0.8}},

μ_B (7)= max { min {0.5, 0.3}, min {0.5, 0.8}, min {0.8, 0.2}, min {0.2, 0.4}},

μ_B (8)=max { min {0.5, 0.0}, min {0.5, 0.2}, min {0.8, 0.8}, min {0.2, 0.0}}} =

= {0.5/5, 0.6/6, 0.5/7, 0.8/8} .

b) **B**(x=2) = {μ_B (5)= { min {0.5, 0.0}, μ_B (6)= { min {0.5, 0.5},

μ_B (7)= { min {0.5, 0.8}, μ_B (8)= { min {0.5, 0.2}} =

$= \{ 0.0/5, 0.5/6, 0.5/7, 0.2/8 \}$.

Note: The result of application of fuzzy composition rule is in all cases "fuzzy set". (As for the variable x described by fuzzy set (**A**) as well as for pro x described for only one symptom (e.g., $x = x_0$)). ■

Generalization of expressions (20.4) – (25.4) gives us very efficient graphic constructions for one symptom ($x=x_0$).

• *Mamdani fuzzy inference for one symptom* $x=x_0$*:*

Let us consider fuzzy sets A, B defined on continuous universes X, Y.

Composition rule (computed for each $y \in Y$, and for $x=x_0$) has the following form:

$$\mu_{BM}(y) = \min \{\mu_A(x_0), \mu_{IMM}(x_0, y)\}\} = \min \{\mu_A(x_0), \min\{\mu_A(x_0), \mu_B(y)\}\}\}. \qquad (26.4)$$

Comparing values of $\mu_A(x_0)$ and $\min \{\mu_A(x_0), \mu_B(y)\}$ we consider these cases:

$$\begin{aligned}
&\text{(i)} \quad (\mu_A(x_0) > \mu_B(y)) \Rightarrow (\mu_{BM}(y) = \mu_B(y)),\\
&\text{(ii)} \quad (\mu_A(x_0) < \mu_B(y)) \Rightarrow (\mu_{BM}(y) = \mu_A(x_0)), \qquad (27.4)\\
&\text{(iii)} \quad (\mu_A(x_0) = \mu_B(y)) \Rightarrow (\mu_{BM}(y) = \mu_A(x_0)).
\end{aligned}$$

See Fig.11.4.

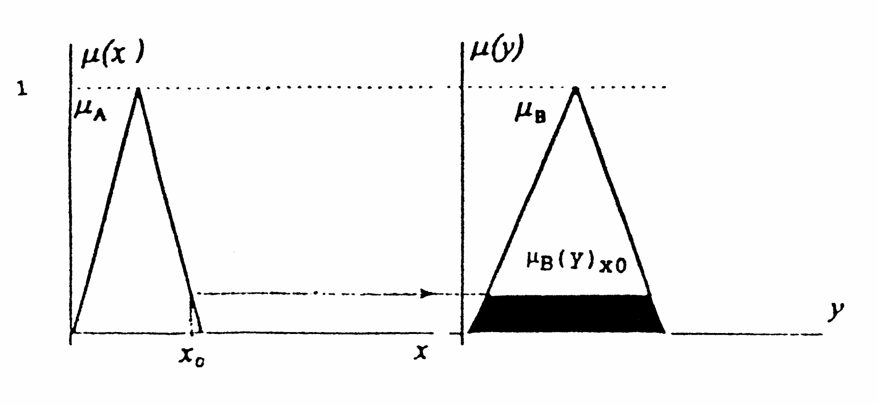

Fig.11.4.

• *Larsen fuzzy inference for value* $x=x_0$*:*

Composition rule for Larsen implication (computed for each $y \in Y$ and for $x = x_0$) has the following form:

$$\mu_{BLA}(y) = \min \{\mu_A(x_0), \mu_{IMLA}(x_0, y)\}\} = \min \{\mu_A(x_0), \mu_A(x_0) * \mu_B(y)\}. \qquad (28.4)$$

In case $L = \langle 0,1 \rangle$ holds:

$$\mu_{BLA}(y) = \mu_A(x_0) * \mu_B(y).$$

The construction for $\mu_{BLA}(y)$ is in Fig. 12.

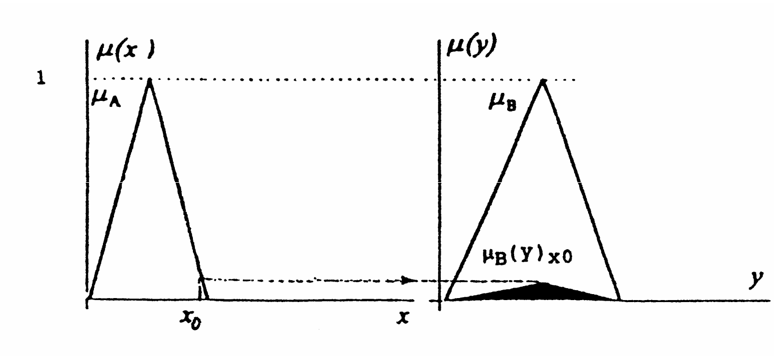

Fig.12.4

Note: There are other forms of implications and inferences named according to their authors – e.g., Gödel, Kleene-Dienes, Zadeh. In detail, e.g., in [2].
Complete conversion between implication and inference satisfies only implication and inference according to Gödel.

Problems and tasks:

1) Which operation on classical sets corresponds to fuzzy composition rule ?
2) Compute fuzzy inference according to Mandani for elements (x, 0.55), (y, 0.35).
3) Verify conversion of fuzzy implication and fuzzy inference according to Lukasiewicz for elements (x, 0.55), (y, 0.35).

Recommended literature:

[1] Kruse, R., Gebhardt, J. and Klawonn, F.: Foundation of Fuzzy Systems. John Wiley & Sons Ltd., Chichester, England, 1994.

[2] Bíla, J., Šmíd, J., Král, F. a Hlaváč, V.: Informační technologie. Databázové a znalostní systémy. ČVUT v Praze, 2009.